

Fine-Wilf graphs and the generalized Fine-Wilf theorem

Stuart A. Rankin

June 9, 2009

Abstract

In 1962, R. C. Lyndon and M. P. Shützenberger established that for any positive integers r and s , any sequence of length at least $r + s$ that is both r -periodic and s -periodic is then (r, s) -periodic. Shortly thereafter (1965), N. J. Fine and H. S. Wilf proved that for any positive integers r and s , if a is an infinite sequence of period r and b is an infinite sequence of period s such that $a_i = b_i$ for all i with $1 \leq i \leq r + s - (r, s)$, then $a = b$. This is equivalent to the following result, which is commonly referred to as the Fine-Wilf theorem: for any positive integers r and s , if w is a finite sequence that is both r -periodic and s -periodic, and $|w| \geq r + s - (r, s)$, then w is (r, s) -periodic. Fine and Wilf also asserted that this bound is best possible, in the sense that for any positive integers r and s , there exists a word w of length $r + s - (r, s) - 1$ that is both r -periodic and s -periodic, but not (r, s) -periodic. This sharpness result has since been established, and these extremal sequences are now much studied. Among other results, it is known that for a given r and s , there is a unique (up to relabelling) sequence of length $r + s - (r, s) - 1$ that is both r -periodic and s -periodic, but not (r, s) -periodic, and in this sequence, exactly two distinct entries appear.

The Fine-Wilf theorem was generalized to finite sequences with three periods by M. G. Castelli, F. Mignosi, and A. Restivo. They introduced a function f from the set of all ordered triples of nonnegative integers to the set of positive integers with the property that if w is a finite sequence with periods p_1 , p_2 , and p_3 , and $|w| \geq f(p)$, where $p = (p_1, p_2, p_3)$, then w is (p) -periodic as well. They further established a condition on p under which the bound $f(p)$ is best possible. In support of their work, they introduced the graphs that we shall refer to as Fine-Wilf graphs. The work of Castelli et al. was generalized by J. Justin, and more broadly by R. Tijdeman and L. Zamboni, who introduced a function fw from the set of all sequences of nonnegative integers to the set of positive integers, and they proved that for a sequence $p = (p_1, p_2, \dots, p_n)$, a finite sequence w with periods p_i , $i = 1, 2, \dots, n$ and length at least $fw(p)$ must be (p) -periodic as well, and that there exists a sequence w of length $fw(p) - 1$ that is p_i -periodic for all i , but not (p) -periodic.

In this paper, we follow ideas introduced by S. Constantinescu and L. Ilie to obtain an alternative formulation of f and fw , and we use

this formulation to establish important properties of f and fw , obtaining in particular new upper and lower bounds for each. We also begin an investigation of Fine-Wilf graphs for arbitrary finite sequences with a view to understanding how the graph may be used to better understand f and fw .

1 Introduction

For any positive integer r , a finite sequence $w = (a_1, a_2, \dots, a_n)$ is said to have period r , or to be r -periodic, if for every positive integer i for which $i, i+r \leq n$, $a_i = a_{i+r}$. In 1962, R. C. Lyndon and M. P. Shützenberger [5] established that for any positive integers r and s , if w is both r -periodic and s -periodic, and $|w| \geq r+s$, then w is $\gcd(r, s)$ -periodic. Shortly thereafter (1965), N. J. Fine and H. S. Wilf [3] proved that for any positive integers r and s , if $\{a_i\}$ is an infinite sequence of period r and $\{b_i\}$ is an infinite sequence of period s such that $a_i = b_i$ for all i with $1 \leq i \leq r+s - \gcd(r, s)$, then $a_i = b_i$ for all i . This is equivalent to the following result, which is commonly referred to as the Fine-Wilf theorem: for any positive integers r and s , if w is a finite sequence that is both r -periodic and s -periodic, and $|w| \geq r+s - \gcd(r, s)$, then w is $\gcd(r, s)$ -periodic. It was also asserted in [3] that this bound is best possible, in the sense that for any positive integers r and s , there exists a word w of length $r+s - \gcd(r, s) - 1$ that is both r -periodic and s -periodic, but not $\gcd(r, s)$ -periodic. This sharpness result has since been established, and these extremal sequences are now much studied. Among other results, it is known that for a given r and s , there is a unique (up to relabelling) sequence of length $r+s - \gcd(r, s) - 1$ that is both r -periodic and s -periodic, but not $\gcd(r, s)$ -periodic, and in this sequence, exactly two distinct entries appear. For example, for $r = 2$ and $s = 3$, the sequence is $(0, 1, 0)$.

Nearly thirty-five years later (1999), the Fine-Wilf theorem was generalized to finite sequences with three periods by M. G. Castelli, F. Mignosi, and A. Restivo [1]. They introduced a function f from the set of all ordered triples of nonnegative integers to the set of positive integers with the property that if w is a finite sequence with periods p_1 , p_2 , and p_3 , and $|w| \geq f(p)$, where $p = (p_1, p_2, p_3)$, then w is $\gcd(p)$ -periodic as well. They further established a condition on p under which the bound $f(p)$ is best possible. The sequences p that met this condition were precisely those for which the unique (up to relabelling) finite sequence of greatest length and with the greatest possible number of distinct entries that had periodicity p_1 , p_2 , and p_3 , but not $\gcd(p_1, p_2, p_3)$ had exactly three distinct entries. In support of their work, they introduced the graphs that we shall refer to as Fine-Wilf graphs $G(p_1, p_2, p_3, n)$, where p_1 , p_2 , and p_3 are distinct nonnegative integers, n is a positive integer, and $G(p_1, p_2, p_3, n)$ denotes the graph with vertex set $\{1, 2, \dots, n\}$, and edge set

$$\{\{i, j\} \mid |i - j| \in \{p_1, p_2, p_3\}\}.$$

The work of Castelli et al. was followed immediately (2000) by work of J.

Justin [4], who extended the definition of the function f to all finite sequences of nonnegative integers, with analogous results.

A broader generalization of the work of Castelli et al. was then given by R. Tijdeman and L. Zamboni [6] (2003). They introduced a function, which we shall denote as fw , from the set of all sequences of nonnegative integers to the set of positive integers, and they proved that for a sequence $p = (p_1, p_2, \dots, p_n)$, a finite sequence w with periods p_i , $i = 1, 2, \dots, n$ and length at least $fw(p)$ must be $\gcd(p)$ -periodic as well, and that there exists a sequence w of length $fw(p) - 1$ that is p_i -periodic for all i , but not $\gcd(p)$ -periodic. At nearly the same time (2005), and independently of the work of Tijdeman and Zamboni, S. Constantinescu and L. Ilie [2] described what amounts to an extension of the function f of Castelli et al., and used f to compute a related function that gives the best bound in all cases. Of course, this related function is the function fw , but the evalution of fw as described by Constantinescu and Ilie is quite different from that of Tijdeman and Zamboni.

In this paper, we establish important properties of the functions f and fw . In particular, we introduce new upper and lower bounds for f . We also begin an investigation of Fine-Wilf graphs for arbitrary p_1, p_2, \dots, p_n , with a view to understanding how the graph depends on the values p_1, p_2, \dots, p_n .

2 Generalization of the Fine-Wilf theorem

Let $\text{OFS}(\mathbb{Z}^+)$ denote the set of all strictly increasing finite sequences of positive integers. For $p \in \text{OFS}(\mathbb{Z}^+)$, let $\gcd(p)$ denote the greatest common divisor of the entries in p , let $|p|$ denote the length of p , and for $1 \leq i \leq |p|$, let p_i denote the i^{th} entry of p and let $p|_i$ denote the truncated sequence (p_1, p_2, \dots, p_i) . Finally, let $\max(p)$ and $\min(p)$ denote p_n , respectively p_1 , where $n = |p|$.

Definition 1 Let $R: \text{OFS}(\mathbb{Z}^+) \rightarrow \text{OFS}(\mathbb{Z}^+)$ denote the function defined as follows. For $p \in \text{OFS}(\mathbb{Z}^+)$, $R(p) = p$ if $|p| = 1$. If $n = |p| > 1$, then form the sequence $(p_2 - p_1, p_3 - p_1, \dots, p_n - p_1) \in \text{OFS}(\mathbb{Z}^+)$, and, if p_1 does not appear in this sequence, insert p_1 so the result is an element of $\text{OFS}(\mathbb{Z}^+)$. The sequence that results (of length either $n - 1$ or n) is denoted by $R(p)$. Moreover, we shall define $p^{(i)} \in \text{OFS}(\mathbb{Z}^+)$ for $i \geq 0$ as follows: $p^{(0)} = p$, and for $k \geq 0$, $p^{(k+1)} = R(p^{(k)})$.

Note that for any $p \in \text{OFS}(\mathbb{Z}^+)$, $\gcd(p) = \gcd(R(p))$.

Definition 2 Define $f: \text{OFS}(\mathbb{Z}^+) \rightarrow \mathbb{Z}^+$ by induction on $\max(p)$ as follows. If $p \in \text{OFS}(\mathbb{Z}^+)$ has $\max(p) = 1$, then $p = (1)$, and we define $f((1)) = 1$. Then for $p \in \text{OFS}(\mathbb{Z}^+)$ with $\max(p) > 1$, define

$$f(p) = \begin{cases} p_1 & \text{if } |p| = 1 \\ p_1 + f(R(p)) & \text{if } |p| > 1. \end{cases}$$

Moreover, the column of sequences whose i^{th} row is $p^{(i)}$, $i \geq 0$, and whose last row is $p^{(m)}$, where m is least subject to the requirement that $|p^{(m)}| = 1$ shall be called the tableau for the calculation of $f(p)$.

For example, if $p = (4, 7)$ or $(4, 7, 9)$, then $f(p) = 10$, as can be seen from the tableaux below.

$\begin{array}{c} 4,7 \\ 3,4 \\ 1,3 \\ 1,2 \\ 1 \end{array}$ <p style="text-align: center;">Tableau for the calculation of $f(p)$ for $p = (4, 7)$.</p>	$\begin{array}{c} 4,7,9 \\ 3,4,5 \\ 1,2,3 \\ 1,2 \\ 1 \end{array}$ <p style="text-align: center;">Tableau for the calculation of $f(p)$ for $p = (4, 7, 9)$.</p>
---	--

Lemma 3 For any $p \in \text{OFS}(\mathbb{Z}^+)$, $f(p) \geq \max(p)$. Furthermore, if $|p| > 1$, then $f(p) \geq 2p_1$.

Proof. By induction on $\max(p)$. It is certainly true when $p = (1)$, and this is the base case $\max(p) = 1$. Suppose now that $m > 1$ is an integer such that the result holds for $p \in \text{OFS}(\mathbb{Z}^+)$ with $1 \leq \max(p) < m$, and let $p \in \text{OFS}(\mathbb{Z}^+)$ be such that $\max(p) = m$. If $|p| = 1$, then $f(p) = p_1 = \max(p)$, so the result holds trivially. Suppose that $|p| > 1$. Then $f(p) = p_1 + f(R(p))$. By hypothesis, $f(R(p)) \geq \max(R(p))$. If $\max(p) - p_1 > p_1$, then $\max(R(p)) = \max(p) - p_1$, otherwise $\max(R(p)) = p_1$. In the former case, we have $f(p) = p_1 + f(R(p)) \geq p_1 + \max(p) - p_1 = \max(p) > 2p_1$, while in the latter case, we have $f(p) = p_1 + f(R(p)) \geq p_1 + p_1 = 2p_1$, and this case occurs when $\max(p) - p_1 \leq p_1$, or $2p_1 \geq \max(p)$. Thus in either case, we have $f(p) \geq \max(p)$, and (recall that $|p| > 1$ in these cases), we have $f(p) \geq 2p_1$. The result follows now by induction. \blacksquare

Corollary 4 For $p \in \text{OFS}(\mathbb{Z}^+)$ with $|p| > 1$, $f(R(p)) \geq p_1$.

Proof. By Lemma 3, $f(p) = p_1 + f(R(p)) \geq 2p_1$, so $f(R(p)) \geq p_1$. \blacksquare

Definition 5 Define $fw: \text{OFS}(\mathbb{Z}^+) \rightarrow \mathbb{Z}^+$ as follows. Let $p \in \text{OFS}(\mathbb{Z}^+)$. Then $fw(p) = fw(p|_{|p|-1})$ if $|p| > 1$, $\gcd(p|_{|p|-1}) = \gcd(p)$, and $\max(p) \geq f(p|_{|p|-1})$, otherwise $fw(p) = f(p)$.

We shall show later (see Proposition 41) that if $p \in \text{OFS}(\mathbb{Z}^+)$ with $|p| > 1$, $\gcd(p|_{|p|-1}) = \gcd(p)$, and $\max(p) < fw(p|_{|p|-1})$, then $fw(p) \leq fw(p|_{|p|-1})$.

Proposition 6 Let $p \in \text{OFS}(\mathbb{Z}^+)$ with $p_1 \neq \gcd(p)$. Then $fw(p) \geq 2p_1$.

Proof. Since $\gcd(p) \neq p_1$, $|p| \geq 2$. If $fw(p) = f(p)$, then the result follows from Lemma 3. Suppose that $fw(p) \neq f(p)$. Then (since $\gcd(p) \neq p_1$), there exists an index $i > 1$ such that $fw(p) = f(p|_i)$, and by Lemma 3, $f(p|_i) \geq 2p_1$. \blacksquare

Proposition 7 Let $p \in \text{OFS}(\mathbb{Z}^+)$, $d = \gcd(p)$, and $\frac{p}{d} = (\frac{p_1}{d}, \dots, \frac{p_{|p|}}{d})$. Then $\frac{p}{d} \in \text{OFS}(\mathbb{Z}^+)$, $f(p) = d f(\frac{p}{d})$, and $fw(p) = d fw(\frac{p}{d})$.

Proof. By induction on $\max(p)$. For $p \in \text{OFS}(\mathbb{Z}^+)$ such that $\max(p) = 1$, it must be that $p = (1)$ and $d = 1$, so the result holds in this case. Suppose now that $p \in \text{OFS}(\mathbb{Z}^+)$ with $\max(p) > 1$ and that the result holds for all sequences in $\text{OFS}(\mathbb{Z}^+)$ with smaller maximum entry. If $|p| = 1$, then $d = p_1$, $p = (p_1)$, and $p/d = (1)$, so $d f(p/d) = p_1 = f((p_1)) = f(p)$ and $d fw(p/d) = p_1 = fw((p_1)) = fw(p)$. Suppose that $n = |p| > 1$, so that $n = |p/d|$ as well. Since there is nothing to prove if $d = 1$, suppose that $d > 1$. We have $f(p) = p_1 + f(R(p))$, and since $\gcd(R(p)) = \gcd(p)$, it follows from our induction hypothesis that $f(R(p)) = d f(R(p)/d) = d f(R(p/d))$ and thus $f(p) = p_1 + d f(R(p/d)) = d(p_1/d + f(R(p/d))) = d f(p/d)$. As for fw , $fw(p) = f(p)$ if $\gcd(p) \neq \gcd(p|_{n-1})$, or $\gcd(p) = \gcd(p|_{n-1})$ and $p_n \geq f(p|_{n-1})$, otherwise $fw(p) = fw(p|_{n-1})$. Observe that $\gcd(p) = \gcd(p|_{n-1})$ if and only if $\gcd(p/d) = 1 = \gcd(p|_{n-1}/d) = \gcd((p/d)|_{n-1})$. Suppose first that $\gcd(p) \neq \gcd(p|_{n-1})$, in which case we have $\gcd(p/d) = 1 \neq \gcd(p|_{n-1}/d) = \gcd((p/d)|_{n-1})$ and thus $fw(p) = f(p)$ and $fw(p/d) = f(p/d)$, so $fw(p) = f(p) = d f(p/d) = d fw(p/d)$. Now suppose that $\gcd(p) = \gcd(p|_{n-1})$, so $\gcd(p/d) = \gcd((p/d)|_{n-1})$. If $p_n < f(p|_{n-1})$, then $fw(p) = f(p) = d f(p/d)$ and $p_n/d < f((p/d)|_{n-1})$, so $d fw(p/d) = d f(p/d) = f(p) = fw(p)$ in this case. Finally, suppose that $p_n \geq f(p|_{n-1})$, so $fw(p) = fw(p|_{n-1})$, and $p_n/d \geq f((p/d)|_{n-1})$. Thus $fw(p/d) = fw((p/d)|_{n-1})$. Since $\gcd(p|_{n-1}) = d > 1$, $\max(p/d) = p_n/d < p_n = \max(p)$, and so we may apply the induction hypothesis to $(p/d)|_{n-1}$ to obtain $fw(p|_{n-1}) = d fw((p/d)|_{n-1})$. Thus $fw(p) = fw(p|_{n-1}) = d fw((p/d)|_{n-1}) = d fw(p/d)$, as required. Since in each case we have $fw(p) = d fw(p/d)$, the result follows by induction. \blacksquare

The next result gives an important lower bound for $f(p)$, and this result can be viewed in a sense as a generalization of the Fine-Wilf theorem. We will later obtain an upper bound (see Proposition 15, also Proposition 39) for $f(p)$ and the combination of that upper bound with the following lower bound, when applied in the case of $|p| = 2$, will give the Fine-Wilf theorem.

Proposition 8 Let $p \in \text{OFS}(\mathbb{Z}^+)$ with $|p| > 1$. Then

$$f(p) \geq \frac{-\gcd(p) + \sum_{i=1}^{|p|} p_i}{|p| - 1},$$

and if equality holds, but for some i , $p_i = 2p_1$, then $i = |p|$ and $f(p) = 2p_1 = \max(p)$.

Proof. By Proposition 7, it suffices to prove the result only for p with $\gcd(p) = 1$. We prove by induction on $\max(p)$ that $|p| > 1$ and $\gcd(p) = 1$

implies $f(p) \geq \frac{-1 + \sum_{i=1}^{|p|} p_i}{|p| - 1}$. The base case, $\max(p) = 1$, is trivially true, so suppose that $p \in \text{OFS}(\mathbb{Z}^+)$ has $\max(p) > 1$ and that the result holds for all elements of $\text{OFS}(\mathbb{Z}^+)$ with smaller maximum entry. Further suppose that $|p| > 1$ and $\gcd(p) = 1$, and let $n = |p|$. Consider first the case when $p_i \neq 2p_1$ for every $i = 1, 2, \dots, n$. Then $|R(p)| = |p| = n > 1$, $\gcd(R(p)) = \gcd(p) = 1$, and the n entries of $R(p)$ are $p_1, p_2 - p_1, p_3 - p_1, \dots, p_n - p_1$ (with p_1 not necessarily in the correct position). We may apply the induction hypothesis to obtain that

$$f(p) = p_1 + f(R(p)) \geq p_1 + \frac{-1 + p_1 + \sum_{i=2}^n (p_i - p_1)}{n - 1} = \frac{-1 + \sum_{i=1}^{|p|} p_i}{|p| - 1},$$

as required. Suppose now that for some j , $p_j = 2p_1$. Then $|R(p)| = |p| - 1 = n - 1$ and $R(p) = (p_2 - p_1, \dots, p_{j-1} - p_1, p_1, p_{j+1} - p_1, \dots, p_n - p_1)$. If $n = 2$, then since $\gcd(p) = 1$, $p = (1, 2)$ and $f(p) = 2$, while $(1 + 2 - 1)/1 = 2$, so the result holds in this case. Note that in this case we have equality, and $f(p) = 2p_1 = \max(p)$, as required. Otherwise, $n > 2$, so $|R(p)| = n - 1 > 1$, $\gcd(R(p)) = \gcd(p) = 1$, and we may apply the induction hypothesis to obtain that

$$f(p) \geq p_1 + \frac{-1 + p_1 + \sum_{i=2, i \neq j}^n (p_i - p_1)}{n - 2} = \frac{-1 - 2p_1 + \sum_{i=1}^n p_i}{n - 2}.$$

Now $\frac{-1 - 2p_1 + \sum_{i=1}^n p_i}{n - 2} \geq \frac{-1 + \sum_{i=1}^n p_i}{n - 1}$ if and only if

$$-1 + \sum_{i=2, i \neq j}^n p_i \geq (2n - 5)p_1.$$

Suppose that $-1 + \sum_{i=2, i \neq j}^n p_i \geq (2n - 5)p_1$. Then $f(p) \geq \frac{-1 - 2p_1 + \sum_{i=1}^n p_i}{n - 2} \geq \frac{-1 + \sum_{i=1}^n p_i}{n - 1}$. Further suppose that $f(p) = \frac{-1 + \sum_{i=1}^n p_i}{n - 1}$. Then $\frac{-1 - 2p_1 + \sum_{i=1}^n p_i}{n - 2} = \frac{-1 + \sum_{i=1}^n p_i}{n - 1}$ and so $\frac{-1 + \sum_{i=1}^n p_i}{n - 1} = 2p_1$. But then $2p_1 = f(p) \geq p_n \geq p_j = 2p_1$, so $p_j = p_n$ and thus $j = n$, as required.

Now suppose that $-1 + \sum_{i=2, i \neq j}^n p_i < (2n - 5)p_1$. Then we have $-1 + \sum_{i=1}^n p_i = -1 + 3p_1 + \sum_{i=2, i \neq j}^n p_i \leq (2n - 2)p_1$, and so $\frac{-1 + \sum_{i=1}^n p_i}{n - 1} \leq 2p_1$. By Lemma 3, $f(p) \geq 2p_1$, so we have $f(p) \geq 2p_1 \geq \frac{-1 + \sum_{i=1}^n p_i}{n - 1}$. Moreover, if $f(p) = \frac{-1 + \sum_{i=1}^n p_i}{n - 1}$, then $f(p) = 2p_1$ and as before, $2p_1 = f(p) \geq p_n \geq p_j = 2p_1$, and so $j = n$. This completes the proof of the inductive step and so the result follows. \blacksquare

Proposition 9 *Let $p \in \text{OFS}(\mathbb{Z}^+)$ with $\min(p) = \gcd(p)$. Then $fw(p) = \min(p)$ and $f(p) = \max(p)$.*

Proof. By induction on $|p|$. If $|p| = 1$, then $fw(p) = f(p) = p_1 = \min(p) = \max(p)$. Suppose now that $n = |p| > 1$. Since $\gcd(p) = p_1 = \gcd(p|_{n-1})$

and by hypothesis, $f(p|_{n-1}) = p_{n-1} < p_n$, we have $fw(p) = fw(p|_{n-1}) = p_1$, while $f(p) = p_1 + f(R(p))$. Since every term is a multiple of p_1 , $\max(R(p)) = \max(p) - p_1$, so by hypothesis, we have $f(p) = p_1 + \max(p) - p_1 = \max(p)$. ■

Proposition 10 *For any $p \in \text{OFS}(\mathbb{Z}^+)$, the following hold.*

1. $f(p) \geq fw(p)$.
2. *If $|p| \geq 2$, and i is such that $1 \leq i < |p|$ and $\gcd(p) = \gcd(p|_j)$ for all $i \leq j < |p|$, and $p_{i+1} \geq f(p|_i)$, then $f(p|_j) = p_j$ for all j with $i+1 \leq j \leq |p|$.*

Proof. By induction on $\max(p)$. Let $p \in \text{OFS}(\mathbb{Z}^+)$. If $\max(p) = 1$, then $p = (1)$ and so $fw(p) = f(p) = 1$. Suppose now that $p \in \text{OFS}(\mathbb{Z}^+)$ has $\max(p) > 1$ and the result holds for all elements of $\text{OFS}(\mathbb{Z}^+)$ with smaller maximum entry. If $|p| = 1$, then $fw(p) = f(p)$, so we may suppose that $n = |p| \geq 2$. If $\gcd(p) \neq \gcd(p|_{n-1})$ or $\gcd(p) = \gcd(p|_{n-1})$ but $p_n < f(p|_{n-1})$, then $fw(p) = f(p)$ by definition, so we may suppose that $\gcd(p) = \gcd(p|_{n-1})$ and $p_n \geq f(p|_{n-1})$. Let i be such that $\gcd(p) = \gcd(p|_j)$ for all $i \leq j < n = |p|$, and $p_{i+1} \geq f(p|_i)$. Furthermore, we may assume without loss of generality that i is minimal in this regard. If $i = 1$, then $p_1 = \gcd(p) = fw(p)$, and thus the result follows from Proposition 9. Consider now the case when $i > 1$. Suppose first that $i < n-1$. Then by the inductive hypothesis applied to $p|_{n-1}$, $f(p|_j) = p_j$ for all $i+1 \leq j < n$. If $f(p) = p_n$, then, since by our induction hypothesis, $f(p|_{n-1}) = p_{n-1} < p_n$, we will have by definition that $fw(p) = fw(p|_{n-1})$ and so $f(p) = p_n > f(p|_{n-1}) \geq fw(p|_{n-1}) = fw(p)$, as required. It therefore suffices to prove that $f(p) = p_n$. In this case, $n > i > 1$, $n-1 > 1$, and so we may apply Lemma 3 to conclude that $f(p) \geq 2p_1$ and thus $p_n > f(p|_{n-1}) \geq 2p_1$. It follows that $p_n - p_1 > p_1$. Now, $f(p) = p_1 + f(R(p))$ and $p_n - p_1 > p_1$, so $\max(R(p)) = p_n - p_1$. Furthermore, we have $p_n - p_1 > p_{n-1} - p_1 = f(p|_{n-1}) - p_1 = f(R(p|_{n-1}))$. Since $\max(R(p)) = p_n - p_1 > p_1$, we have $R(p)|_{|R(p)|-1} = R(p|_{n-1})$, so

$$\max(R(p)) = p_n - p_1 > f(R(p|_{n-1})) = f(R(p)|_{|R(p)|-1}).$$

Since $\max(R(p)) < \max(p)$, our induction hypothesis applies to $R(p)$ and since

$$|R(p)| \geq 2, \quad \gcd(R(p)) = \gcd(p) = \gcd(p|_{n-1}) = \gcd(R(p|_{n-1}))$$

and $\max(R(p)) > f(R(p)|_{|R(p)|-1})$, we conclude that $f(R(p)) = \max(R(p)) = p_n - p_1$. Thus $f(p) = p_1 + f(R(p)) = p_1 + p_n - p_1 = p_n$.

It remains to consider the case when $i = n-1$. We have $p_n \geq f(p|_{n-1}) = p_1 + f(R(p|_{n-1}))$. Furthermore, we have $\gcd(p) = \gcd(p|_{n-1})$. We wish to prove that $f(p) = p_n$ (and thus $f(p) \geq fw(p|_{n-1}) = fw(p)$ as well). Since $n > 1$, we have

by Lemma 3 that $p_n \geq f(p|_{n-1}) \geq 2p_1$, and so $p_n - p_1 \geq p_1$. Suppose first that $p_n - p_1 > p_1$. Then as above, we have $R(p) = (p_2 - p_1, p_3 - p_1, \dots, p_1, \dots, p_n - p_1)$, and so $R(p)|_{|R(p)|-1} = R(p|_{n-1})$. It then follows that

$$\max(R(p)) = p_n - p_1 > f(R(p|_{n-1})) = f(R(p)|_{|R(p)|-1}).$$

Then by our induction hypothesis (since $\gcd(R(p)) = \gcd(p) = \gcd(p|_{n-1}) = \gcd(R(p|_{n-1}))$), we have $f(R(p)) = \max(R(p)) = p_n - p_1$, which yields $f(p) = p_1 + f(R(p)) = p_1 + p_n - p_1 = p_n$. Finally, consider the case when $p_n - p_1 = p_1$, or $p_n = 2p_1$. In this case, we have $R(p) = (p_2 - p_1, p_3 - p_1, \dots, p_{n-1} - p_1, p_1) = R(p|_{n-1})$. As well, from $2p_1 = p_n \geq f(p|_{n-1}) \geq 2p_1$, we obtain that $f(p|_{n-1}) = 2p_1$, and thus $p_1 + f(R(p|_{n-1})) = 2p_1$, or $f(R(p|_{n-1})) = p_1$. Now, $f(p) = p_1 + f(R(p)) = p_1 + f(R(p|_{n-1})) = p_1 + p_1 = 2p_1 = p_n$, as required. ■

In particular, if $p \in \text{OFS}(\mathbb{Z}^+)$ has $n = |p| > 1$, $\gcd(p|_{n-1}) = \gcd(p)$, and $p_n \geq f(p|_{n-1})$, then $f(p) = p_n$.

Definition 11 Let $p \in \text{OFS}(\mathbb{Z}^+)$. We say that p is trim if either $|p| = 1$, or else $n = |p| > 1$ and either $\gcd(p) \neq \gcd(p|_{n-1})$ or else $\gcd(p) = \gcd(p|_{n-1})$ but $\max(p) < f(p|_{n-1})$. For any $p \in \text{OFS}(\mathbb{Z}^+)$, there exists i with $1 \leq i \leq |p|$ such that $p|_i$ is trim, and $q \in \text{OFS}(\mathbb{Z}^+)$ is called the trimmed form of p if $q = p|_i$ where i is maximal with respect to the property $p|_i$ is trim.

We note that even if p is trim, there may exist i with $1 < i < |p|$ such that $p|_i$ is not trim.

Corollary 12 Let $p \in \text{OFS}(\mathbb{Z}^+)$. If p is not trim, then $f(p) = \max(p)$.

Proof. Since $|p| = 1$ implies that p is trim, we have $n = |p| > 1$. Since $\gcd(p) = \gcd(p|_{n-1})$ and $p_n \geq f(p|_{n-1})$, we may take $i = n - 1$ in Proposition 10 (ii) to obtain that $f(p) = p_n = \max(p)$. ■

For $p \in \text{OFS}(\mathbb{Z}^+)$ with $n = |p| \geq 2$, if $\gcd(p) = \gcd(p|_{n-1})$ and $\max(p) \geq fw(p|_{n-1})$, then we shall say that $p|_{n-1}$ is obtained by trimming p . Evidently, for any $p \in \text{OFS}(\mathbb{Z}^+)$, we may iteratively apply the trimming operation to obtain $q = p|_j$ for some $j > 1$ with q trim, and we note that $fw(p) = fw(q)$.

Lemma 13 Let $p \in \text{OFS}(\mathbb{Z}^+)$. If p is trim, then $fw(p) = f(p)$. Furthermore, if $|p| > 1$, then $\min(p) > \gcd(p)$.

Proof. That $fw(p) = f(p)$ is immediate from the definition of fw . If $|p| > 1$, then by Proposition 9, we have $\min(p) > \gcd(p)$. ■

Proposition 14 Let $p \in \text{OFS}(\mathbb{Z}^+)$. If p is trim with $|p| > 1$, then $f(p) > \max(p)$. In addition, if $R(p)$ is not trim, then $f(p) = 2p_1$.

Proof. The proof is by induction on $\max(p)$. If $\max(p) = 1$, then $p = (1)$ and the implication holds trivially ($|p| > 1$ fails). Suppose now that $\max(p) > 1$ and that the result holds for every trim element of $\text{OFS}(\mathbb{Z}^+)$ with smaller maximum entry. If $|p| = 1$, then again the implication holds trivially. Suppose that $n = |p| > 1$. If $|R(p)| = 1$, then $R(p) = (\gcd(p))$, and thus $p = (\gcd(p), 2\gcd(p))$, which is not trim. Thus $|R(p)| > 1$. Suppose first that $R(p)$ is trim. Then by our inductive hypothesis, $f(R(p)) > \max(R(p))$. If $p_n \geq 2p_1$, then $\max(R(p)) = p_n - p_1$ and thus $f(p) = p_1 + f(R(p)) > p_1 + p_n - p_1 = p_n$. Now consider the case when $p_n < 2p_1$. Then by Corollary 4, $f(p) = p_1 + f(R(p)) \geq p_1 + p_1 = 2p_1 > p_n$. Thus if $R(p)$ is trim, then $f(p) > p_n$. Suppose now that $R(p)$ is not trim. Then by Proposition 10 (ii), $f(R(p)) = \max(R(p))$. If $p_n < 2p_1$, then $\max(R(p)) = p_1 > p_n - p_1$, and then $f(p) = p_1 + f(R(p)) = 2p_1 > p_n$, as required. Suppose next that $p_n > 2p_1$. Then $\max(R(p)) = p_n - p_1$, $|R(p)| = n$, and $R(p)|_{n-1} = R(p)|_{n-1}$. Since $R(p)$ is not trim, we have $\gcd(R(p)) = \gcd(R(p)|_{|R(p)|-1})$ and $\max(R(p)) \geq f(R(p)|_{|R(p)|-1})$, so

$$\gcd(p) = \gcd(R(p)) = \gcd(R(p)|_{n-1}) = \gcd(R(p)|_{n-1}) = \gcd(p|_{n-1})$$

and $p_n - p_1 = \max(R(p)) \geq f(R(p)|_{n-1}) = f(R(p)|_{n-1})$. Since p is trim and $|p| > 1$, this implies that $p_n < f(p|_{n-1}) = p_1 + f(R(p)|_{n-1}) \leq p_1 + p_n - p_1 = p_n$, which is impossible. Thus if $R(p)$ is not trim, but p is trim, it is not possible that $p_n > 2p_1$. Finally, suppose that $p_n = 2p_1$. Then $\gcd(p) = \gcd(p|_{n-1})$, so since p is trim, we have $p_n < f(p|_{n-1})$. Now $p_n - p_1 = p_1$, so $R(p) = (p_2 - p_1, \dots, p_{n-1} - p_1, p_1) = R(p|_{n-1})$, so $f(R(p)) = f(R(p|_{n-1}))$, and thus

$$f(p) = p_1 + f(R(p)) = p_1 + f(R(p|_{n-1})) = f(p|_{n-1}) > p_n.$$

As well, $\max(R(p)) = p_1$, and so by Corollary 12, $f(p) = p_1 + f(R(p)) = p_1 + \max(R(p)) = 2p_1$. This completes the proof of the inductive step. ■

The following result gives an upper bound for f that is reminiscent of the Fine-Wilf theorem. Later (see Proposition 39), we shall establish a generalization of this which for p trim with $|p| \geq 3$ offers a slightly improved upper bound for fw .

Proposition 15 *For $p \in \text{OFS}(\mathbb{Z}^+)$, $f(p) \leq \min(p) + \max(p) - \gcd(p)$.*

Proof. By Proposition 7, it suffices to prove that if $\gcd(p) = 1$, then $f(p) \leq \min(p) + \max(p) - 1$. The proof is by induction on $\max(p)$, with the base case $\max(p) = 1$, so $p = (1)$ and $f(p) = 1 = \min(p) + \max(p) - 1$. Suppose now that $\gcd(p) = 1$ and $\max(p) > 1$ and the result holds for every element of $\text{OFS}(\mathbb{Z}^+)$ with smaller maximum entry. If $|p| = 1$, then $p = (\gcd(p)) = (1)$ and so $\max(p) = 1$, which is not the case. Thus $n = |p| > 1$. Since $\gcd(R(p)) = \gcd(p) = 1$, we may apply the induction hypothesis to $R(p)$ to obtain that $f(p) = p_1 + f(R(p)) \leq p_1 + \min(R(p)) + \max(R(p)) - 1$. We consider three

cases. The first occurs when $p_1 \leq p_2 - p_1$, in which case $\min(R(p)) = p_1$, and $\max(R(p)) = p_n - p_1$. We have $f(p) \leq p_1 + p_1 + p_n - p_1 - 1 = p_1 + p_n - 1$, as required. Next, suppose that $p_2 - p_1 < p_1 \leq p_n - p_1$. Then $\min(R(p)) = p_2 - p_1$, $\max(R(p)) = p_n - p_1$, and we note that $p_2 < 2p_1$. We have $f(p) \leq p_1 + p_2 - p_1 + p_n - p_1 - 1 = p_2 - p_1 + p_n - 1 < 2p_1 - p_1 + p_n - 1 = p_1 + p_n - 1$. Finally, suppose that $p_n - p_1 < p_1$, so that $\min(R(p)) = p_2 - p_1$ and $\max(R(p)) = p_1$. Then $f(p) \leq p_1 + p_2 - p_1 + p_1 - 1 = p_1 + p_2 - 1 \leq p_1 + p_n - 1$. This completes the proof of the inductive step. \blacksquare

Corollary 16 For $p \in \text{OFS}(\mathbb{Z}^+)$, $fw(p) \leq \min(p) + \max(p) - \gcd(p)$.

Proof. By Proposition 10, $fw(p) \leq f(p)$, and by Proposition 15, $f(p) \leq \min(p) + \max(p) - \gcd(p)$. \blacksquare

Theorem 17 (Fine-Wilf) Let $p \in \text{OFS}(\mathbb{Z}^+)$ be trim with $|p| = 2$. Then $fw(p) = p_1 + p_2 - \gcd(p)$.

Proof. Since p is trim, $fw(p) = f(p)$, and by Proposition 8, $f(p) \geq p_1 + p_2 - \gcd(p)$, while by Proposition 15, $f(p) \leq p_1 + p_2 - \gcd(p)$. \blacksquare

3 The Fine-Wilf graphs $G(p, k)$

Definition 18 Let $p \in \text{OFS}(\mathbb{Z}^+)$. For any $k \in \mathbb{Z}^+$, $G(p, k)$ shall denote the simple graph with vertex set $\{1, \dots, k\}$ and edge set

$$\{\{i, j\} \mid |i - j| = p_k \text{ for some } k = 1, 2, \dots, |p|\}.$$

The values $k = fw(p)$ and $k = fw(p) - 1$ feature prominently in the development of the theory, and we shall let G_p and G'_p denote $G(p, fw(p))$ and $G(p, fw(p) - 1)$, respectively.

Note that if $p, q \in \text{OFS}(\mathbb{Z}^+)$ and q is the trimmed form of p , then $fw(p) = fw(q)$, and $G_p = G_q$, $G'_p = G'_q$.

Our first goal in this section is to establish that for $p \in \text{OFS}(\mathbb{Z}^+)$, the graph G_p has exactly $d = \gcd(p)$ connected components, each isomorphic to $G_{p/d}$. For example, for $p = (6, 8, 10)$, $fw(p) = 12$, $\gcd(p) = 2$, and G_p has two connected components, each isomorphic to the connected graph $G_{(3,4,5)}$, where by Proposition 7, $fw(3, 4, 5) = fw(6, 8, 10)/2 = 6$.

$$\begin{array}{c} 4 - 1 - 6 - 3 \\ | \quad | \\ 5 - 2 \end{array} \qquad \begin{array}{c} 7 - 1 - 11 - 5 \\ | \quad | \\ 9 - 3 \end{array} \qquad \begin{array}{c} 8 - 2 - 12 - 6 \\ | \quad | \\ 10 - 4 \end{array}$$

$$G_{(3,4,5)}$$

$$G_{(6,8,10)}$$

The second major objective of the section is then to establish that G'_p has more than d components.

Definition 19 Let $p \in \text{OFS}(\mathbb{Z}^+)$ with $|p| > 1$. For any $k \geq 1$, the function $\alpha_{p,k} : G(R(p), k) \rightarrow G(p, p_1 + k)$ is defined by $\alpha_{p,k}(i) = p_1 + i$.

We note that in general, $\alpha_{p,k}$ is not a graph homomorphism. However, it does have the following important property.

Lemma 20 *Let $p \in \text{OFS}(\mathbb{Z}^+)$ with $|p| > 1$, and let $k \geq 1$. If i and j belong to the same connected component of $G(R(p), k)$, then $\alpha_{p,k}(i)$ and $\alpha_{p,k}(j)$ belong to the same connected component of $G(p, p_1 + k)$.*

Proof. Let i and j be such that $1 \leq i < j \leq k$ and $\{i, j\}$ is an edge in $G(R(p), k)$. Suppose first of all that $j - i = p_1$. Then $\{j, i\}$ is an edge in $G(p, p_1 + k)$, and since $\{i, \alpha_{p,k}(i)\}$ and $\{j, \alpha_{p,k}(j)\}$ are also edges in $G(p, p_1 + k)$, it follows that $\alpha_{p,k}(i)$ and $\alpha_{p,k}(j)$ are connected in $G(p, p_1 + k)$. Now suppose that $j - i = p_t - p_1$ for some t with $2 \leq t \leq \max(p)$. Then $\{\alpha_{p,k}(j), i\}$ is an edge in $G(p, p_1 + k)$. Since $\{i, \alpha_{p,k}(i)\}$ is an edge in $G(p, p_1 + k)$, it follows that $\alpha_{p,k}(i)$ and $\alpha_{p,k}(j)$ belong to the same connected component of $G(p, p_1 + k)$. ■

Thus if C is a connected component of $G(R(p), k)$, then $\alpha_{p,k}(C)$ is contained in a component of $G(p, p_1 + k)$.

We are now ready to demonstrate that for any $p \in \text{OFS}(\mathbb{Z}^+)$ with $\gcd(p) = 1$, $G_{p/d}$ is connected. In fact, we have the following slightly stronger result.

Proposition 21 *Let $p \in \text{OFS}(\mathbb{Z}^+)$ with $\gcd(p) = 1$, and let $k \geq \text{fw}(p)$. Then $G(p, k)$ is connected.*

Proof. The proof is by induction on $\max(p)$. The base case occurs when $\max(p) = 1$ (since $\gcd(1) = 1$), and in this case, $\text{fw}(p) = 1$. Since for any $k \geq \text{fw}(p) = 1$, $G(p, k)$ is a chain of length $k - 1$, the result holds when $\max(p) = 1$. Suppose now that $p \in \text{OFS}(\mathbb{Z}^+)$ has $\gcd(p) = 1$ and $\max(p) > 1$ and that the result holds for all elements of $\text{OFS}(\mathbb{Z}^+)$ of smaller maximum entry and greatest common divisor equal to 1. Note that $\gcd(p) = 1$ and $\max(p) > 1$ imply that $n = |p| > 1$.

Case 1: p is not trim. Then $\gcd(p) = \gcd(p|_{n-1})$, $\text{fw}(p|_{n-1}) = \text{fw}(p) \leq f(p) = p_n$, so $G(p, \text{fw}(p)) = G(p, \text{fw}(p|_{n-1})) = G(p|_{n-1}, \text{fw}(p|_{n-1}))$. Since $\gcd(p) = 1$, we have $\gcd(p|_{n-1}) = 1$ and so by hypothesis, $G(p|_{n-1}, \text{fw}(p|_{n-1}))$ is connected, which establishes that $G(p, \text{fw}(p))$ is connected. Suppose now that $k \geq \text{fw}(p)$. If $p_1 > 1$, then by Proposition 6, $\text{fw}(p) \geq 2p_1 > p_1$, while if $p_1 = 1$, then by Proposition 9, $\text{fw}(p) = p_1$. In any event, we have $\text{fw}(p) \geq p_1$. Thus for any vertex i in $G(p, k)$ with $i > \text{fw}(p) \geq p_1$, we may write $i = qp_1 + j$ where $1 \leq j \leq p_1$, so that j is a vertex of $G(p, \text{fw}(p))$ and i is connected to j in $G(p, k)$. Thus $G(p, k)$ is connected.

Case 2: p is trim. In this case, $\text{fw}(p) = f(p)$, so $G(p, \text{fw}(p)) = G(p, f(p)) = G(p, p_1 + f(R(p)))$. Now, $1 = \gcd(p) = \gcd(R(p))$ and $\max(R(p)) < \max(p)$, and since $f(R(p)) \geq \text{fw}(R(p))$, it follows from the induction hypothesis that $G(R(p), f(R(p)))$ is connected. Thus

$$\{p_1 + 1, p_1 + 2, \dots, p_1 + f(R(p))\},$$

the image of $G(R(p), f(R(p)))$ under $\alpha_{R(p), f(R(p))}$, is a connected subset of $G(p, p_1 + f(R(p))) = G(p, f(p))$. Now, for any i with $1 \leq i \leq p_1$, i is connected to $i + p_1 \leq 2p_1 \leq fw(p)$ in $G(p, f(p))$, so $G(p, f(p))$ is connected. Since $G(p, f(p)) = G(p, fw(p))$, $G(p, fw(p))$ is connected. Suppose now that $k \geq fw(p)$. Then $G(p, fw(p))$ is a connected subgraph of $G(p, k)$. Since p is trim and $|p| > 1$, $fw(p) = f(p) \geq 2p_1$, so for i in $G(p, k)$ with $i > fw(p) \geq 2p_1$, we may write $i = qp_1 + j$ where $1 \leq j \leq p_1$. Thus j is a vertex of $G(p, fw(p))$ and i is connected to j in $G(p, k)$, which proves that $G(p, k)$ is connected. ■

Proposition 22 *Let $p \in \text{OFS}(\mathbb{Z}^+)$, $k \in \mathbb{Z}^+$. If i and j belong to the same connected component of $G(p, k)$, then $\gcd(p)$ divides $i - j$.*

Proof. It suffices to observe that if i and j are adjacent in $G(p, k)$, then $|i - j| = p_k$ for some k with $1 \leq k \leq |p|$. Then since $\gcd(p)$ divides p_k , it follows that $\gcd(p)$ divides $i - j$. ■

Corollary 23 *For $p \in \text{OFS}(\mathbb{Z}^+)$ and $k \in \mathbb{Z}^+$ with $k \geq \gcd(p)$, $G(p, k)$ has at least $\gcd(p)$ connected components.*

Proof. If $1 \leq i < j \leq \gcd(p)$, then it follows from Proposition 22 that i and j can't be in the same connected component of $G(p, k)$. ■

Proposition 24 *Let $p \in \text{OFS}(\mathbb{Z}^+)$, and let $d = \gcd(p)$. Then for each $i = 1, 2, \dots, d$, the map $\gamma_i: G_{p/d} \rightarrow G_p$ defined by $\gamma_i(j) = i + (j - 1)d$ for $1 \leq j \leq fw(p/d)$ is an injective graph homomorphism which is an isomorphism from $G_{p/d}$ onto the subgraph $\gamma_i(G_{p/d})$ of G_p . Moreover, G_p has exactly d components, the images of γ_i , $i = 1, 2, \dots, d$; that is, the congruence classes of the interval $\{1, 2, \dots, fw(p)\}$.*

Proof. It is immediate from Proposition 22 that each component of G_p is contained in the image of γ_i for some i with $1 \leq i \leq d$, and by Proposition 21 $G_{p/d}$ is connected. It remains only to prove that for each such i , γ_i is a graph homomorphism, injectivity being obvious. Let j, k be vertices of $G_{p/d}$. Since $|\gamma_i(j) - \gamma_i(k)| = |(j - 1)d - (k - 1)d| = |j - k|d$, it follows that $|i - j| = p_t/d$ if and only if $|\gamma_i(j) - \gamma_i(k)| = p_t$. Thus γ_i is a graph isomorphism from $G_{p/d}$ onto the subgraph $\gamma_i(G_{p/d})$ of G_p . ■

Corollary 25 *Let $p \in \text{OFS}(\mathbb{Z}^+)$ and let $k \geq fw(p)$. Then $G(p, k)$ has exactly $d = \gcd(p)$ components, the congruence classes of the interval $\{1, 2, \dots, k\}$ modulo d .*

Proof. Since G_p is a subgraph of $G(p, k)$, and by Proposition 24, G_p has exactly $\gcd(p)$ components, it suffices to prove that each vertex i of $G(p, k)$ is connected to a vertex in the subgraph G_p . But $i = qp_1 + j$, where $1 \leq j \leq p_1$, and j is a vertex of G_p , with i connected to j by a path of length q in $G(p, k)$. ■

Proposition 26 *Let $p \in \text{OFS}(\mathbb{Z}^+)$ have $|p| > 1$, and let $k \geq 1$. Then $\alpha_{p,k}$ induces an injective map from the set of components of $G(R(p), k)$ into the set of components of $G(p, p_1 + k)$.*

Proof. Suppose not, and of all pairs of disconnected elements of $G(R(p), k)$ whose images are connected in $G(p, p_1 + k)$, choose two, say $i < j$, whose shortest path joining their images in $G(p, p_1 + k)$ has least length. Let $i_0 = i + p_1, i_1, \dots, i_n = j + p_1$ denote a shortest path in $G(p, p_1 + k)$ from $i + p_1$ to $j + p_1$. Let $m \geq 0$ be the maximum index such that for all t with $0 \leq t \leq m$, $i_t > p_1$. Suppose that $i_1 > p_1$. If $i_0 > i_1$, then $i_0 - i_1 = p_t$ for some t , and thus $i - i_1 = p_t - p_1$. But then $i \geq i_1 > p_1$, and thus $(i - p_1) - (i_1 - p_1) = p_t - p_1$, which means that $\{i - p_1, i_1 - p_1\}$ is an edge in $G(R(p), k)$. Since $\{i, i - p_1\}$ would also be an edge in $G(R(p), k)$, this would imply that i and $i_1 - p_1$ are in the same component of $G(R(p), k)$, and thus $i_1 - p_1$ and j are in different components of $G(R(p), k)$. Since $(i_1 - p_1) + p_1 = i_1$ and $j + p_1$ are connected in $G(p, k + p_1)$ by a path of length $n - 1$, we have a contradiction to the minimality of n . Thus if $i_1 > p_1$, it must be that $i_0 < i_1$ (since $i_0 = i_1$ is not possible). In this case, $i_1 - i_0 = p_t$ for some t , and so $(i_1 - p_1) - i = p_t$. Thus $i_1 - p_1 > p_t \geq p_1$, and so $i_1 - 2p_1 > 0$, from which we obtain that $(i_1 - 2p_1) - i = p_t - p_1 \geq 0$. If $(i_1 - 2p_1) - i = 0$, then $i_1 - 2p_1 = i$, otherwise $\{i_1 - 2p_1, i\}$ is an edge in $G(R(p), k)$. Since $i_1 \leq k + p_1$, we have $i_1 - p_1 \leq k$ and thus $\{i_1 - p_1, i_1 - 2p_1\}$ is an edge in $G(R(p), k)$ as well. Thus $i_1 - p_1$ lies in the same component of $G(R(p), k)$ as does i , which means that $i_1 - p_1$ and j lie in different components of $G(R(p), k)$, again contradicting the minimality of n . Thus $i_1 \leq p_1$. We consider two cases: $i_1 \leq k$, and $i_1 > k$. Suppose first that $i_1 \leq k$. Since $i_1 \leq p_1$, we have $i + p_1 > i_1$, and so $(i + p_1) - i_1 = p_t$ for some t . But then $i - i_1 = p_t - p_1 \geq 0$, and since $i, i_1 \leq k$, with $i + p_1 = i_0 \neq i_1$, it follows that either $i = i_1$ or else $\{i, i_1\}$ is an edge in $G(R(p), k)$. Now since $i_n = j + p_1 > p_1$, while $i_1 \leq p_1$, we conclude that $n \geq 2$. If $i_2 \leq p_1$, then $|i_1 - i_2| < p_1$, contradicting the fact that $|i_1 - i_2| = p_r$ for some r . Thus $i_2 > p_1 \geq i_1$, and so $i_2 - i_1 = p_r$ for some r . Now $p_1 < i_2 \leq k + p_1$, so $0 < i_2 - p_1 \leq k$ and $(i_2 - p_1) - i_1 = p_r - p_1$, so $\{(i_2 - p_1), i_1\}$ is an edge in $G(R(p), k)$. Thus $i_2 - p_1$ and i lie in the same component of $G(R(p), k)$, contradicting the minimality of n . Thus $i_1 \leq k$ is impossible, which means that we must have $i_1 > k$. But then $i \leq k < i_1$. However, since $i_1 < i + p_1$ and $\{i_1, i + p_1\}$ is an edge in $G(R(p), k)$, we have $(i + p_1) - i_1 = p_r$ for some r . But then $i - i_1 = p_r - p_1 \geq 0$ and thus $i \geq i_1$, impossible. It follows therefore that the map on components that is induced by $\alpha_{p,k}$ is injective. ■

Proposition 27 *Let $p \in \text{OFS}(\mathbb{Z}^+)$. Then $\min(p) \neq \gcd(p)$ implies that G'_p has more than $\gcd(p)$ components.*

Proof. As usual, the proof is by induction on $\max(p)$, and the result is trivially true for $\max(p) = 1$. Suppose then that $p \in \text{OFS}(\mathbb{Z}^+)$ has $\max(p) > 1$ and the result holds for all elements of $\text{OFS}(\mathbb{Z}^+)$ with smaller maximum entry. Let $d = \gcd(p)$ and $n = |p|$, and suppose that $p_1 \neq d$. We first consider

the case when p is not trim. In this case, it follows from Corollary 12 that $p_n = f(p) \geq fw(p|_{n-1}) = fw(p)$, so $G'_p = G'_{p|_{n-1}}$. The inductive hypothesis can therefore be applied to conclude that $G'_{p|_{n-1}}$ has more than $\gcd(d) = \gcd(p|_{n-1})$ components.

Assume now that p is trim.

Suppose first that $R(p)$ is trim. If $\min(R(p)) = \gcd(R(p))$, then by Lemma 13, $|R(p)| = 1$ and so $R(p) = (d)$ since $d = \gcd(p) = \gcd(R(p))$. But then $p = (d, 2d)$, so $\gcd(p) = \min(p)$, which is not the case. Thus $\min(R(p)) \neq \gcd(R(p))$, and so by the inductive hypothesis, $G'_{R(p)}$ has more than $\gcd(p) = \gcd(R(p))$ components. But $fw(R(p)) = f(R(p)) = f(p) - p_1 = fw(p) - p_1$, and so $G'_{R(p)} = G(R(p), fw(p) - p_1 - 1)$ has more than $\gcd(R(p)) = \gcd(p)$ components. By Proposition 26, $\alpha_{p, fw(p) - 1 - p_1} : G(R(p), fw(p) - 1 - p_1) \rightarrow G(p, fw(p) - 1)$ induces an injective function on components, and so it follows that $G'_p = G(p, fw(p) - 1)$ has more than $\gcd(p)$ components.

Suppose now that $R(p)$ is not trim. Then by Proposition 14, $f(p) = 2p_1$. Since p is trim, we have $fw(p) = f(p) = 2p_1$, and so it follows that $G'_p = G(p, fw(p) - 1) = G(p, 2p_1 - 1)$ has $\{p_1\}$ as a component. By Proposition 26, the map

$$\alpha_{p, p_1 - 1} : G(R(p), p_1 - 1) = G(R(p), fw(p) - p_1 - 1) \rightarrow G(p, fw(p) - 1) = G_p f$$

induces an injective map on components. Since p_1 is not in the image of $\alpha_{p, p_1 - 1}$, G'_p has at least one more component than $G(R(p), p_1 - 1)$. We have $p_1 = f(R(p)) \geq fw(R(p))$. If $fw(R(p)) < p_1$, then $G(R(p), p_1 - 1)$ has $\gcd(R(p)) = \gcd(p)$ components, which then implies that G'_p has more than $\gcd(p)$ components. It remains to consider the case when $fw(R(p)) = p_1$. Then $G(R(p), p_1 - 1) = G(R(p), fw(R(p)) - 1) = G'_{R(p)}$. Suppose that $\min(R(p)) = \gcd(R(p))$. Then by Proposition 9, $p_1 = fw(R(p)) = \min(R(p)) = \gcd(R(p)) = \gcd(p)$, which is not the case. Thus $\min(R(p)) \neq \gcd(R(p))$ and we may therefore apply the induction hypothesis to $G'_{R(p)} = G(R(p), p_1 - 1)$, to conclude that $G(R(p), p_1 - 1)$ has more than $\gcd(R(p)) = \gcd(p)$ components, which then implies that $G(p, fw(p) - 1)$ has more than $\gcd(p)$ components. This completes the proof of the inductive step. \blacksquare

Note: Let $m(p) = fw(p) - 1$. The relationship between $m(p)$ and $m(p/d)$, $d = \gcd(p)$ is quite straightforward. By Proposition 7, $fw(p) = dfw(p/d)$, so $m(p) = fw(p) - 1 = dfw(p/d) - 1 = d(fw(p/d) - 1 + 1) - 1 = d(m(p/d) + 1) - 1$.

Proposition 28 *Let $p \in \text{OFS}(\mathbb{Z}^+)$. For each positive integer k , the map $\tau_{p,k} : G(p, k) \rightarrow G(p, k)$ given by $\tau_{p,k}(i) = k - (i - 1)$ is a graph automorphism of order 2.*

Proof. For $i \in \mathbb{Z}$, $1 \leq i \leq k$ if and only if $-1 \geq -i \geq -k$, which in turn holds if and only if $k \geq k - i + 1 \geq 1$, so i is a vertex of $G(p, k)$ if and only if $k - (i - 1)$ is a vertex of $G(p, k)$. Thus $\tau_{p,k}$ is a function from the vertex set of $G(p, k)$ to itself, evidently of order 2. Furthermore, for $1 \leq i < j \leq k$, $j - i = p_t$

for some t if and only if $\tau_{p,k}(i) - \tau_{p,k}(j) = (k - i + 1) - (k - j + 1) = j - i = p_t$, and so $\{i, j\}$ is a edge in $G(p, k)$ if and only if $\{\tau_{p,k}(i), \tau_{p,k}(j)\}$ is an edge in $G(p, k)$. \blacksquare

Of course, since $\tau_{p,k}$ is a graph automorphism of $G(p, k)$, it induces a permutation of the set of components of $G(p, k)$, and in general, this permutation is nontrivial. For example, $p = (8, 12, 14)$ is a trim sequence, so $fw(p) = f(p) = 16$, and $\tau_{p,fw(p)}: G_p \rightarrow G_p$ induces a nontrivial permutation on the set consisting of the two components of G_p . This may be quickly verified by observing that since $\tau_{p,fw(p)}(i) = 17 - i$ for each i , we have in particular that $\tau_{p,fw(p)}(6) = 17 - 6 = 11$.

$$\begin{array}{ccccccc} 6 & -14 & -2 & -16 & -4 & -12 & \\ | & | & & & & & \\ 10 & 8 & & & & & \end{array} \quad \begin{array}{ccccccc} 11 & -3 & -15 & -1 & -13 & -5 & \\ | & | & & & & & \\ 7 & 9 & & & & & \end{array}$$

$$G((8, 12, 14), 16)$$

More generally, we may easily describe the fixed points of the permutation that $\tau_{p,fw(p)}$ induces on the set of components of G_p . By Proposition 24, i and j are in the same component of G_p if and only if $i \equiv j \pmod{d}$, where $d = \gcd(p)$. Since d divides $fw(p)$, it follows that for any i , we have $i - \tau_{p,fw(p)}(i) = 2i - 1 - fw(p)$, so $\tau_{p,fw(p)}$ fixes the component of i (setwise) if and only if d divides $2i - 1$. Thus in general, the permutation on the set of components of G_p that is induced by $\tau_{p,fw(p)}$ will have few fixed points. The following result thus comes as a bit of a surprise.

Proposition 29 *Let $p \in \text{OFS}(\mathbb{Z}^+)$. If $\gcd(p) < p_1$, then $\tau_{p,fw(p)-1}(C) = C$ for each component C of G'_p .*

Proof. We are to prove that for each i with $1 \leq i \leq fw(p) - 1$, i and $\tau_{p,fw(p)-1}(i) = fw(p) - 1 - (i - 1) = fw(p) - i$ are connected in G'_p , and as usual, we shall use induction on $\max(p)$. If $\max(p) = 1$, then $p = (1)$ and $\gcd(p) = 1$, so the result holds vacuously. Suppose now that $\max(p) > 1$, and that the result holds for all elements of $\text{OFS}(\mathbb{Z}^+)$ with maximum entry less than $\max(p)$. If $|p| = 1$, then again, we have $\gcd(p) = p_1$ and so the result holds vacuously. Suppose that $|p| > 1$. Suppose first that p is not trim, and let q denote the trimmed form of p . Since $fw(p) = fw(q)$, $G'_p = G'_q$, $\tau_{p,fw(p)-1} = \tau_{q,fw(q)-1}$, and $\max(q) < \max(p)$, we may apply the induction hypothesis to q , so the result holds for q and thus for p . We may therefore assume that p is trim, so $fw(p) = f(p)$. Let i be such that $1 \leq i \leq f(p) - 1$. Note that if $1 \leq i \leq p_1$, then $f(p) - 1 \geq f(p) - i \geq f(p) - p_1$, and by Lemma 3, $f(p) \geq 2p_1$, so $f(p) - 1 \geq f(p) - i \geq p_1$. Since $\tau_{p,fw(p)-1}$ is an automorphism of order 2, we may assume that $i \geq p_1$.

We apply the induction hypothesis to $R(p)$ to conclude that $i - p_1$ and $fw(R(p)) - (i - p_1)$ are connected in $G'_{R(p)}$. We claim that $i - p_1$ and $f(R(p)) - (i - p_1)$ are connected in $G(R(p), f(R(p)) - 1)$. If $fw(R(p)) = f(R(p))$, there is nothing to show, while if $fw(R(p)) < f(R(p))$, then by Proposition 25, the components of $G(R(p), f(R(p)) - 1)$ are precisely the congruence classes of $\{1, 2, \dots, f(R(p)) - 1\}$ modulo $d = \gcd(R(p)) = \gcd(p)$, and since $f(R(p)) \equiv 0 \equiv$

$f(p) \bmod d$, it follows that $i - p_1 \equiv fw(R(p)) - (i - p_1) \equiv f(R(p)) - (i - p_1) \bmod d$ and so $i - p_1$ and $f(R(p)) - (i - p_1)$ are connected in $G(R(p), f(R(p)) - 1)$, as required. But then by Proposition 20, i and $p_1 + f(R(p)) - i + p_1 = f(p) - (i - p_1) = fw(p) - (i - p_1)$ are connected in $G(p, p_1 + f(R(p)) - 1) = G'_p$. Since $f(p) - i$ and $f(p) - i + p_1$ are connected in G'_p as well, we finally obtain that i and $fw(p) - i$ are connected in G'_p , as required. This completes the proof of the inductive step, and so the result follows. \blacksquare

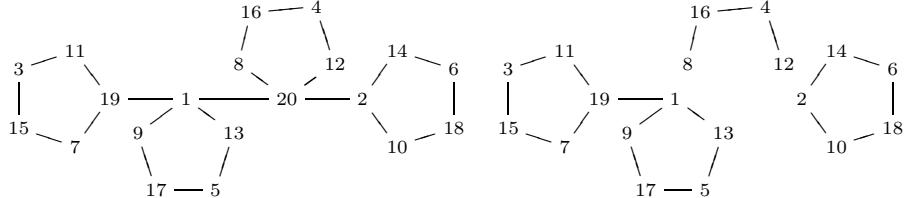
We have seen that for $k > fw(p) - 1$, in particular for $k = fw(p)$, that the permutation on the set of components of $G(p, k)$ by $\tau_{p,k}$ may have many fixed points, and the same holds true for $k < fw(p) - 1$. For example, if $p = (3, 5)$, then $fw(p) = 7$, so $fw(p) - 1 = 6$. Let us consider $k = 5$. The components of $G(p, 5)$ are $\{1, 4\}$, $\{2, 5\}$, and $\{3\}$, and we have $\tau_p(\{1, 4\}) = \{5 - (1 - 1), 5 - (4 - 1)\} = \{2, 5\}$, so $\tau_p(\{2, 5\}) = \{1, 4\}$, while $\tau_p(\{3\}) = \{3\}$.

$$1 - 4 \quad 3 \quad 2 - 5 \quad 3 - 6 - 1 - 4 \quad 2 - 5$$

$$k = 5, \tau_p(i) = 6 - i \quad k = 6, \tau_p(i) = 7 - i$$

We remark that in general, $|\text{Aut}(G_p)| > 2$. For example, if $p = (8, 12, 18, 19)$, then $\text{Aut}(G_p) \simeq (C_2^2 \times C_2^2) \rtimes C_2$, where the action of C_2 on $C_2 \times C_2$ is to swap coordinates, while $\text{Aut}(G'_p) \simeq D_5 \times C_2 \times ((C_2 \times C_2) \rtimes C_2)$, where again, the action of C_2 is to swap coordinates.

Example 30 Let $p = (8, 12, 18, 19)$. Then



$$\text{Aut}(G_p) \simeq (C_2^2 \times C_2^2) \rtimes C_2 \quad \text{Aut}(G'_p) \simeq D_5 \times C_2 \times ((C_2 \times C_2) \rtimes C_2).$$

Proposition 31 Let $p \in \text{OFS}(\mathbb{Z}^+)$ be such that there exist i and j with $1 \leq i < j \leq |p|$ and p_j is a multiple of p_i , and let $q \in \text{OFS}(\mathbb{Z}^+)$ denote the sequence obtained by deleting p_j from p . Then

1. For every positive integer k , the components of $G(p, k)$ are identical to those of $G(q, k)$. In particular, $\kappa_{G(p,k)} = \kappa_{G(q,k)}$.
2. $fw(p) = fw(q)$.
3. $p_j \geq fw(p_{j-1})$, and $\gcd(p|_j) = \gcd(p|_{j-1})$.

Proof. Suppose that $p_j = tp_i$. Let $1 \leq r < s \leq k$, and suppose that r and s are joined by a walk in $G(p, k)$. If no edge in the walk is determined by p_j , then the walk is a walk from r to s in $G(q, k)$. Otherwise, there is at least one edge in the walk that is determined by p_j . If $\{a, b\}$ is an edge in the walk, where $a > b$ and $a - b = p_j$, then we may replace $\{a, b\}$ by the path

$b, b + p_i, b + 2p_i, \dots, b + tp_i = b + p_j = a$. Apply this procedure to all edges in the walk that are determined by p_j to obtain a walk from r to s that does not use any edge determined by p_j ; that is, a walk from r to s in $G(q, k)$. Thus any component of $G(p, k)$ is contained in a component of $G(q, k)$. Since $G(q, k)$ is a spanning subgraph of $G(p, k)$, each component of $G(q, k)$ is contained in a component of $G(p, k)$. Thus the components of $G(p, k)$ are identical to those of $G(q, k)$.

Now since $\gcd(p) = \gcd(q)$, it follows from Corollary 25 and Proposition 27 that if $\gcd(p) \neq p_1$, then

$$\begin{aligned} fw(p) &= \min\{k \mid \kappa_{G(p,k)} = \gcd(p)\} \\ &= \min\{k \mid \kappa_{G(q,k)} = \gcd(p)\} = fw(q), \end{aligned}$$

while if $\gcd(p) = p_1$, then by Proposition 9, we have $fw(p) = p_1 = fw(q)$.

For (iii), we may assume that $\gcd(p) = 1$ and that $j = |p|$. By Proposition 27, either $p_1 = \min(p) = \gcd(p)$ or else G'_p is not connected. If $p_1 = \gcd(p)$, then since by (ii), $fw(p|_{j-1}) = fw(p)$, we have $fw(p|_{j-1}) = p_1 < p_j$. Otherwise, $G'_p = G_p - fw(p)$ is not connected, while G_p is connected, and so $fw(p)$ is a cut-vertex of G_p . Suppose that $p_j < fw(p|_{j-1})$, so that by (ii), $p_j < fw(p)$, and thus $1 + p_j \leq fw(p)$. We have $p_j = cp_i$ for some integer c and some p_i in p , so there is a path from 1 to $1 + p_j$ in G_p consisting of c edges determined by p_1 . But then we may follow the edge from $1 + p_j$ back to 1 that is determined by p_j , and so p_j belongs to a cycle in G_p . Since $p_j \leq fw(p)$, there exists $t \geq 0$ such that $t + p_j = fw(p)$, and then $1 + t, 1 + t + p_i, \dots, 1 + t + cp_i = 1 + t + p_n = fw(p)$, $1 + t$ is a cycle in G_p through $fw(p)$, which contradicts the fact that $fw(p)$ is a cut-vertex of G_p . Thus $p_j \geq fw(p|_{j-1})$. Finally, since p_j is a multiple of p_i , it is immediate that $\gcd(p|_j) = \gcd(p|_{j-1})$. ■

The preceding result suggests that it will be convenient to introduce notation for those sequences with no entry a multiple of another.

Definition 32 For $p \in \text{OFS}(\mathbb{Z}^+)$, the sequence obtained from p by iterated deletion of any term that is a multiple of another shall be called the reduced form of p . Furthermore, we shall say that p is reduced if p is equal to its reduced form.

Corollary 33 If $p \in \text{OFS}(\mathbb{Z}^+)$ and q is the reduced form of p , then $fw(p) = fw(q)$.

Proof. This is immediate from Proposition 31. ■

Lemma 34 Let $p \in \text{OFS}(\mathbb{Z}^+)$, and let k be a positive integer. If the interval $\{1, 2, \dots, p_1\}$ is contained within a component of $G(p, k)$, then $G(p, k)$ is connected.

Proof. If $p_1 = 1$, then $G(p, k)$ is a complete graph, hence connected. Suppose that $p_1 > 1$, in which case the fact that 1 and 2 are connected in $G(p, k)$ implies

that $k > p_1$. For each i with $p_1 < i \leq k$, there exist positive integers m and j such that $i = mp_1 + j$ and $1 \leq j \leq p_1$. Thus there is a path of length m from i to j in $G(p, k)$. It follows now that $G(p, k)$ is connected. \blacksquare

Definition 35 Let $p \in \text{OFS}(\mathbb{Z}^+)$. For j such that $2 \leq j \leq |p|$, p_j is said to be redundant in p if $\gcd(p|_j) = \gcd(p|_{j-1})$ and $p_j \geq f(p|_{j-1})$.

Proposition 36 Let $p \in \text{OFS}(\mathbb{Z}^+)$ and j be such that p_j is redundant in p . If $q \in \text{OFS}(\mathbb{Z}^+)$ denotes the sequence obtained by deleting p_j from p , then $fw(p) = fw(q)$.

Proof. First, note that $\gcd(p) = \gcd(q)$. Let $n = |p|$. If $p_1 = \gcd(p)$, then $fw(p) = p_1 = fw(q)$, so we may suppose that $p_1 \neq \gcd(p)$. We may further assume that $\gcd(p) = 1$, and that p is trim. Suppose that $fw(p) < fw(q)$. By Proposition 27, $G(q, fw(p))$ is not connected, and by Proposition 14, $p_j < fw(p)$. Thus $fw(p|_{j-1}) \leq p_j < fw(p)$, and so $G(p|_{j-1}, fw(p|_{j-1}))$ is a connected subgraph of $G(q, fw(p))$. By Proposition 6 and the fact that $j-1 \geq 2$ (since $j > 1$ and if $j = 2$, then from $\gcd(p|_j) = \gcd(p|_{j-1})$ we would have $\gcd(p) = p_1$), we have $2p_1 \leq fw(p|_{j-1})$, and so the interval $\{1, 2, \dots, p_1\}$ is contained within a component of $G(q, fw(p))$. By Lemma 34, this implies that $G(q, fw(p))$ is connected, which is not the case. Therefore, it must be that $fw(p) \geq fw(q)$. Suppose that $fw(p) > fw(q)$. Then $G(q, fw(q))$ is a connected subgraph of G'_p . By Proposition 6 applied to q , $fw(q) \geq 2p_1$. Thus the interval $\{1, 2, \dots, 2p_1\}$ is contained within a component of G'_p and so by Lemma 34, G'_p is connected. But this is not the case, so we conclude that $fw(p) = fw(q)$, as required. \blacksquare

As a consequence of Proposition 36, we see that given $p \in \text{OFS}(\mathbb{Z}^+)$, we may iteratively remove redundant entries without regard to the order of removal to end up with a sequence with no redundant entries. More precisely, if we construct a list of elements of $\text{OFS}(\mathbb{Z}^+)$ with first entry p , and each subsequent entry obtained by selecting and removing a redundant element from the current entry in the list, then the last entry in the list will equal the element of $\text{OFS}(\mathbb{Z}^+)$ that is obtained from p by identifying all redundant elements in p and removing them all at the same time.

Definition 37 For $p \in \text{OFS}(\mathbb{Z}^+)$, let $r(p)$ equal the number of redundant entries in p , and let \hat{p} denote the element of $\text{OFS}(\mathbb{Z}^+)$ that is obtained by deleting all $r(p)$ redundant entries in p , so $r(p) = |p| - |\hat{p}|$. \hat{p} shall be called the totally reduced form of p , and we shall say that p is totally reduced if $p = \hat{p}$.

Corollary 38 Let $p \in \text{OFS}(\mathbb{Z}^+)$. Then $fw(\hat{p}) = fw(p)$.

Proof. This follows from Proposition 36 and the fact that \hat{p} can be formed from p by $r(p)$ iterations of the process of selecting and removing a redundant entry. \blacksquare

Note that if $p \in \text{OFS}(\mathbb{Z}^+)$ is totally reduced, then for every j with $1 \leq j \leq |p|$, $p|_j$ is both reduced and trim.

We are now in a position to give an upper bound for $fw(p)$ that is an improvement over that given in Proposition 15 (provided that $r(p) < |p| - 1$, its maximum possible value).

Proposition 39 *For each $p \in \text{OFS}(\mathbb{Z}^+)$,*

$$fw(p) \leq \min(p) + \max(p) - \gcd(p)(|p| - 1 - r(p)).$$

Proof. Let $d = \gcd(p)$. By Proposition 7, $r(p) = r(p/d)$, $d \cdot fw(p/d) = fw(p)$, while by definition of p/d , $\min(p) = d \min(p/d)$, $\max(p) = d \max(p/d)$ and $|p| = |p/d|$. It suffices therefore to prove the result for $p \in \text{OFS}(\mathbb{Z}^+)$ with $\gcd(p) = 1$, and this we shall do by induction on $\max(p)$. If $\gcd(p) = 1$ and $\max(p) = 1$, then $p = (1)$ and so $fw(p) = 1 = \min(p) = \max(p) = |p|$, while $r(p) = 0$, so $\min(p) + \max(p) - (|p| - 1 - r(p)) = 2 \geq 1 = fw(p)$.

Suppose now that $\gcd(p) = 1$, $\max(p) > 1$, and the result holds for all elements of $\text{OFS}(\mathbb{Z}^+)$ with greatest common divisor 1 and smaller maximum entry. Since $|p| = 1$ would imply that $1 = \gcd(p) = \max(p) > 1$, it follows that $n = |p| > 1$. Consider \hat{p} , the totally reduced form of p . Since $\min(\hat{p}) = p_1$, $\max(\hat{p}) = p_j$ for some j with $2 \leq j \leq n$, we have $\max(\hat{p}) \leq p_n$. If we are able to prove that $fw(\hat{p}) \leq \min(\hat{p}) + \max(\hat{p}) - (|\hat{p}| - 1)$, then by Corollary 38, $fw(p) = fw(\hat{p}) \leq \min(\hat{p}) + \max(\hat{p}) - (|\hat{p}| - 1) \leq p_1 + p_n - (|p| - r(p) - 1)$, as required. Thus we may assume that p is totally reduced, and we are to prove that $fw(p) \leq p_1 + p_n - (|p| - 1)$. Since p is totally reduced, it is in particular trim, and so $fw(p) = f(p) = p_1 + f(R(p))$. If $R(p)$ is not trim, then by Proposition 14, $f(p) = 2p_1$, and since $p_n - p_1 \geq |p| - 1$, we have $p_1 \leq p_n - (|p| - 1)$ and so $fw(p) = f(p) = 2p_1 \leq p_1 + p_n - (|p| - 1)$, as required. Thus we may assume that $R(p)$ is trim, so $fw(R(p)) = f(R(p))$. Furthermore, by Proposition 31, the fact that p is totally reduced means that p is reduced and so in particular, no entry of p is a multiple of p_1 . Thus $p_1 \neq p_j - p_1$ for every j with $1 \leq j \leq n$, so $|R(p)| = |p|$. Apply the induction hypothesis to $R(p)$ to obtain

$$\begin{aligned} fw(p) &= p_1 + f(R(p)) = p_1 + fw(R(p)) \\ &\leq p_1 + \min(R(p)) + \max(R(p)) - (|R(p)| - 1 - r(R(p))) \\ &= p_1 + \min(R(p)) + \max(R(p)) - (|p| - 1 - r(R(p))). \end{aligned}$$

It will suffice to prove that $\min(R(p)) + \max(R(p)) + r(R(p)) \leq p_n$. Let us first treat the case when $R(p)$ is totally reduced; that is, $r(R(p)) = 0$. There are three subcases to consider. If $p_1 \leq p_2 - p_1$, then $\min(R(p)) = p_1$ and $\max(R(p)) = p_n - p_1$, so $\min(R(p)) + \max(R(p)) + r(R(p)) = p_1 + p_n - p_1 + 0 = p_n$, as required. Suppose now that $p_2 - p_1 < p_1 \leq p_n - p_1$, so that $\min(R(p)) = p_2 - p_1$ and $\max(R(p)) = p_n - p_1$. Then $\min(R(p)) + \max(R(p)) + r(R(p)) = p_2 - p_1 + p_n - p_1 + 0 = p_n + p_2 - 2p_1$. But from $p_2 - p_1 < p_1$, we have $p_2 - 2p_1 < 0$ and so $p_n + p_2 - 2p_1 < p_n$. Finally, suppose that $p_n - p_1 < p_1$, so that $\min(R(p)) = p_2 - p_1$ and $\max(R(p)) = p_1$, which implies that $\min(R(p)) + \max(R(p)) + r(R(p)) = p_2 - p_1 + p_1 = p_2 \leq p_n$, as required.

We now treat the case when $R(p)$ is not totally reduced, so that $r(R(p)) > 0$. Let j be such that $p_j - p_1$ is redundant in $R(p)$. Let S , respectively S' , denote the initial segment of $R(p)$ that consists of the entries up to but not including $p_j - p_1$, respectively the entries up and including $p_j - p_1$, so $p_j - p_1 \geq fw(S)$ and $\gcd(S) = \gcd(S')$. Since p is totally reduced and therefore reduced, $p_1 = p_j - p_1$ is not possible. Consider first the possibility that $p_1 < p_j - p_1$. Then either

$$S = (p_2 - p_1, p_3 - p_1, \dots, p_1, \dots, p_{j-1} - p_1), \text{ and}$$

$$S' = (p_2 - p_1, p_3 - p_1, \dots, p_1, \dots, p_j - p_1),$$

or else

$$S = (p_1, p_2 - p_1, p_3 - p_1, \dots, p_{j-1} - p_1), \text{ and}$$

$$S' = (p_1, p_2 - p_1, p_3 - p_1, \dots, p_j - p_1),$$

depending on whether $p_2 - p_1 < p_1$ or $p_2 - p_1 > p_1$. In either case, we have $S = R(p)|_{j-1} = R(p|_{j-1})$ and $S' = R(p|_j)$. Thus $R(p|_j)$ is not trim, so by Proposition 14, $f(p|_j) = 2p_1$. Now p is totally reduced, so $p|_j$ is trim, and since $J > 1$, Proposition 14 implies that $f(p|_j) > \max(p|_j) = p_j$. Thus $2p_1 > p_j$; equivalently, $p_j - p_1 < p_1$, contradicting our assumption that $p_j - p_1 > p_1$. Thus $p_j - p_1 > p_1$ cannot hold, and since p is reduced, $2p_1 \neq p_j$ and thus we must have $p_j - p_1 < p_1$. Since $j = 2$ would mean that $p_2 - p_1$ is redundant and thus not the minimum entry of $R(p)$, it must be that $j > 2$ and so we have established that if j is any index such that $p_j - p_1$ is redundant in $R(p)$, then $p_2 < p_j < 2p_1$. Thus

$$0 < r(R(p)) \leq |\{j \mid p_2 < p_j < 2p_1\}| + 1,$$

where we add 1 to acknowledge that p_1 might be redundant in $R(p)$. Thus $0 < r(R(p)) \leq (2p_1 - p_2 - 1) + 1 = 2p_1 - p_2$, and so $p_2 < 2p_1$, which means that $p_2 - p_1 = \min(R(p))$. We consider two cases according to whether $p_1 < p_n - p_1$ or $p_1 > p_n - p_1$ (again, $p_1 = p_n - p_1$ is not possible since p is totally reduced).

Case 1: $p_1 < p_n - p_1$. Then $\max(R(p)) = p_n - p_1$, and so $\min(R(p)) + \max(R(p)) + r(R(p)) \leq p_2 - p_1 + p_n - p_1 + 2p_1 - p_2 = p_n$.

Case 2: $p_1 > p_n - p_1$. Then $\max(R(p)) = p_1$, and $p_n < 2p_1$. Recall that $R(p)$ is trim and so $\max(R(p)) = p_1$ is not redundant in $R(p)$. Thus in this case, we have

$$r(R(p)) \leq |\{j \mid p_2 < p_j < 2p_1\}| = |\{3, 4, \dots, n\}| = n - 2.$$

Thus $\min(R(p)) + \max(R(p)) + r(R(p)) \leq p_2 - p_1 + p_1 + n - 2 = p_2 + n - 2 \leq p_n$. This completes the proof of the inductive step, and so the result follows by induction. \blacksquare

Corollary 40 *Let $p \in \text{OFS}(\mathbb{Z}^+)$ be totally reduced. Then $fw(p) \leq \min(p) + \max(p) - \gcd(p)(|p| - 1)$.*

Proposition 41 Let $p \in \text{OFS}(\mathbb{Z}^+)$ be such that $|p| > 1$, $\gcd(p|_{|p|-1}) = \gcd(p)$, and $\max(p) < fw(p|_{|p|-1})$. Then $fw(p|_{|p|-1}) \geq fw(p)$.

Proof. By Proposition 7, it suffices to prove this for $p \in \text{OFS}(\mathbb{Z}^+)$ for which $\gcd(p) = 1$. Note that by Proposition 9, if $\min(p) = 1$, then $fw(p) = 1 = fw(p|_{|p|-1})$, and so the result holds in this case. Thus we may assume that $\min(p) > 1 = \gcd(p)$.

Thus we consider $p \in \text{OFS}(\mathbb{Z}^+)$ such that $|p| > 1$, $\gcd(p|_{|p|-1}) = 1 < \min(p)$, and $\max(p) < fw(p|_{|p|-1})$. Suppose, contrary to our claim, that $fw(p|_{|p|-1}) < fw(p)$. Let $k = fw(p) - 1 - fw(p|_{|p|-1})$. If $k = 0$, then $G'_p = G(p, fw(p|_{|p|-1}))$, which has $G(p|_{|p|-1}, fw(p|_{|p|-1}))$ as a spanning subgraph. By Corollary 25, $G(p|_{|p|-1}, fw(p|_{|p|-1}))$ is connected. Thus if $k = 0$, then G'_p is connected. However, by Proposition 27 (since $\min(p) > 1 = \gcd(p)$), G'_p has more than $\gcd(p) = 1$ components, so we would have a contradiction. Thus $k > 0$. Now by Proposition 39 and the fact that $\max(p) < fw(p|_{|p|-1})$, we have

$$\begin{aligned} k &= fw(p) - 1 - fw(p|_{|p|-1}) \\ &\leq \min(p) + \max(p) - |p| + 1 + r(p) - 1 - \max(p) \\ &= \min(p) - (|p| - r(p)) < \min(p) < \max(p) < fw(p|_{|p|-1}). \end{aligned}$$

But every element of $\{1, 2, \dots, fw(p|_{|p|-1})\}$ is connected to 1 by a path in $G(p|_{|p|-1}, fw(p|_{|p|-1}))$. Since $\max(p) < fw(p|_{|p|-1})$, it follows that every element of $\{1, \dots, fw(p) - 1\}$ is connected to 1 by a path in $G(p|_{|p|-1}, fw(p) - 1)$ and thus in G'_p , which contradicts the fact that G'_p is not connected. ■

Note that it is possible that under the conditions of the preceding result, we may actually have $fw(p|_{|p|-1}) > fw(p)$. The lexically first such example would be $p = (5, 7, 8)$, where $fw(p) = 10$, while $fw(5, 7) = 11$.

It might be tempting to believe that fw grows monotonically with respect to the product order on sequences of a given length and greatest common divisor 1, and, as the Fine-Wilf theorem tells us, this is indeed the case for sequences of length 2. However, this observation does not hold even for sequences of length 3. For example, $fw(7, 9, 11) = 15$, while $fw(7, 9, 13) = 14$.

4 Combinatorics on words

Let $p \in \text{OFS}(\mathbb{Z}^+)$ with $\gcd(p) = 1$ and $|p| > 1$, and consider the tableau for the computation of $f(p)$. Let m be minimum subject to the requirement that $|p^{(m)}| = 1$. Then $p^{(m)} = (1)$, and $p^{(m-1)} = (1, 2)$. For each i with $0 \leq i \leq m$, we shall call $p^{(i)}$ a jump if $f(p^{(i)}) = 2p_1^{(i)}$, and in the tableau for the computation of $f(p)$, we shall prefix each jump with a plus sign (+). Furthermore, let $J(p)$

denote the number of jumps in the tableau for the calculation of $f(p)$. For example, $p = (6, 10, 13)$ has tableau

$$\begin{array}{r} 6,10,13 \\ + \quad 4,6,7 \\ + \quad 2,3,4 \\ + \quad 1,2 \\ 1 \end{array}$$

and so $J(p) = 3$. We observe that $p^{(m)}$ is never a jump, while $p^{(m-1)}$ is always a jump. For each $i = 1, \dots, m$, let G^i denote the graph $G(p^{(m-i)}, f(p^{(m-i)}) - 1)$, so that G^1 is the null graph on a single vertex, and $G^m = G(p, f(p) - 1)$. Now if $m > 1$; equivalently, $p \neq (1)$, for each $i = 2, \dots, m$, let $\alpha_i: G^{i-1} \rightarrow G^i$ denote $\alpha_{p^{(m-(i-1))}, f(p^{(m-(i-1))})-1}$ (see Definition 19). This makes sense as

$$G(R(p^{(m-i)}), f(p^{(m-i+1)}) - 1) = G(p^{(m-i+1)}, f(p^{(m-i+1)}) - 1) = G^{i-1}$$

and

$$G(p^{(m-i)}, p_1^{(m-i)} + f(p^{(m-i+1)}) - 1) = G(p^{(m-i)}, f(p^{(m-i)}) - 1) = G^i$$

and $\alpha_{p^{(m-(i-1))}, f(p^{(m-(i-1))})-1}$ is a function from $G(R(p^{(m-i)}), f(p^{(m-i+1)}) - 1)$ to $G(p^{(m-i)}, p_1^{(m-i)} + f(p^{(m-i+1)}) - 1)$. Thus, for a vertex j , $\alpha_i(j) = p_1^{(m-i)} + j$. By Proposition 26, for each i , α_i induces an injective map from the set of components of G^{i-1} into the set of components of G^i , and G^0 has a single component. Moreover, the image of α_i is the set

$$\{p_1^{(m-i)} + 1, \dots, p_1^{(m-i)} + f(p^{(m-i+1)}) - 1\} = \{p_1^{(m-i)} + 1, \dots, f(p^{(m-i)}) - 1\}.$$

If $f(p^{(m-i)}) > 2p_1^{(m-i)}$, then for each $k \in \{1, 2, \dots, p_1^{(m-i)}\}$, $k + p_1^{(m-i)} \leq 2p_1^{(m-i)} \leq f(p^{(m-i)}) - 1$, and $\{k, k + p_1^{(m-i)}\}$ is an edge in G^i joining k to a vertex in the image of α_i , so G^i and G^{i-1} have exactly the same number of components. On the other hand, if $f(p^{(m-i)}) = 2p_1^{(m-i)}$, then $p_1^{(m-i)}$ has degree 0 in G^i , so $\{p_1^{(m-i)}\}$ is a component of G^i that is not contained in the image of α_i . For any k with $1 \leq k < p_1^{(m-i)}$, $\{k, k + p_1^{(m-i)}\}$ is an edge in G^i joining k to a vertex in the image of α_i , and so the number of components of G^i is one more than the number of components of G^{i-1} . This proves the following result.

Proposition 42 *Let $p \in \text{OFS}(\mathbb{Z}^+)$ with $|p| > 1$ and $\text{gcd}(p) = 1$. Then the number of components of $G(p, f(p) - 1)$ is equal to $J(p)$, the number of jumps in the tableau for the computation of $f(p)$.*

Now, for $p = (p_1, p_2, \dots, p_n)$ trim, $G(p, f(p) - 1) = G'_p$, and the word w of length $f(p) - 1$ formed by labelling the k components of G'_p using, say, the integers from 0 to $k - 1$, then setting w_i equal to the label of the component containing vertex i of G'_p has periods p_1, p_2, \dots, p_n , but not $\text{gcd}(p)$. By Proposition 42, the number of distinct letters in the word is equal to $J(p)$. Moreover, by Proposition 29, w is a palindrome.

We observe that the preceding discussion also shows that w can be calculated from the tableau for the calculation of $f(p)$. We begin at row $p^{(m-1)}$ with word 0. Then at stage $p^{(i)}$, shift the preceding word $p_1^{(i)}$ spaces to the right. If $p^{(i)}$ is not a jump, then the preceding word has length at least $p_1^{(i)}$ and we fill in the first $p_1^{(i)}$ locations of the new word with the first $p_1^{(i)}$ entries in the preceding word, while if $p^{(i)}$ is a jump, then the preceding word has length $p_1^{(i)} - 1$, and we fill in the first $p_1^{(i)} - 1$ spaces with the first $p_1^{(i)} - 1$ entries of the preceding word and then introduce a new symbol for the vertex at position $p(i)_1$.

For example, $p = (6, 10, 13)$ has tableau

$$\begin{array}{rcc}
 6,10,13 & 0102010102010 \\
 + & 4,6,7 & 0102010 \\
 + & 2,3,4 & 010 \\
 + & 1,2 & 0 \\
 & & 1
 \end{array}$$

and we have shown the construction of the word $w = 0102010102010$. Note that $J(6, 10, 13) = 3$ and indeed, w has three distinct letters.

References

- [1] M. G. Castelli, F. Mignosi, A. Restivo, Fine and Wilf's theorem for three periods and a generalization of Sturmian words, *Theor. Comput. Sci.* 218(1999), 83-94.
- [2] S. Constantinescu and L. Ilie, Generalised Fine and Wilf's theorem for arbitrary number of periods, *Theor. Comput. Sci.*, 339(2005), 49-60.
- [3] N. J. Fine and H. S. Wilf, Uniqueness theorems for periodic functions, *Proc. Amer. Math. Soc.*, 16(1965) 109–114.
- [4] J. Justin, On a paper by Castelli, Mignosi, Restivo, *Theoret. Informatics Appl.*, 34(2000), 373–377.
- [5] Lyndon, R. C. and Shutzerberger, M. P. The equation $a^m = b^n c^p$ in a free group, *Michigan Math. J.*, 9(1962), 289–298.
- [6] R. Tijdeman and L. Zamboni, Fine and Wilf words for any periods, *Indag. Math.*, 14(2003), 135–147.